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Fusion products, Kostka polynomials and fermionic characters of $\widehat{su}(r+1)_k$

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Abstract
Using a form factor approach, we define and compute the character of the fusion product of rectangular representations of $\widehat{su}(r+1)$. This character decomposes into a sum of characters of irreducible representations, but with q-dependent coefficients. We identify these coefficients as (generalized) Kostka polynomials. Using this result, we obtain a formula for the characters of arbitrary integrable highest weight representations of $\widehat{su}(r+1)$ in terms of the fermionic characters of the rectangular highest weight representations.

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1. Introduction

In an earlier paper [1] (which we will refer to as I) we provided a physics application of the ideas of Feigin and Stoyanovsky (FS) [2], who showed that by counting the number of linearly independent symmetric polynomials that vanish when $k+1$ of their variables coincide one may obtain fermionic formulae [3] for the characters of integrable representations of the affine Lie algebra $\widehat{su}(2)_k$ [4]. We also sketched how a similar count of the number of symmetric polynomials in two types of variables allows the computation of the character of the vacuum representation of $\widehat{su}(3)_k$. The reason for the restriction to the vacuum representation of $\widehat{su}(3)_k$ was that a naïve application of the FS strategy to most representations of higher rank groups does not yield the characters. Here we will explain both why the naïve method fails, and what must be done to correct it. To be more precise, we explain the ideas behind the corrected character formulae in a form accessible to physicists. We will not give all the technical details of the proofs. The mathematical details and the proofs are given in [5].

The basic idea of [2] is most simply explained in the language of conformal field theory: we compute matrix elements of ladder-operator currents between the highest weight state that defines the representation of interest and any other weight state in the representation. These matrix elements are rational functions (in the $\widehat{su}(2)$ case they are symmetric polynomials) in the co-ordinates of the current operators and have certain restrictions imposed on them by relations in the Lie algebra, and by the highest weight condition. By counting the number of possible functions satisfying these constraints, we are able to count the dimension of all weight subspaces.

For $\widehat{su}(r+1)$, and for a representation whose top graded component forms a finite-dimensional representation of $su(r+1)$ having only one non-zero Dynkin index (and hence corresponds to a rectangular Young diagram), it is not difficult to count the dimensions of the function space. For a general representation, it is hard. We find, however, that by introducing a fusion-product representation we can retain a function space whose dimensions we can count. The price to be paid is that the fusion-product representation is reducible. The generating function for the dimensions, although of fermionic form, is therefore a sum of characters of irreducible representations, moreover one with coefficients that are polynomials in the variable $1/q$. These q-dependent coefficients can be identified as being generalized Kostka polynomials [6, 7]. The decomposition may then be inverted to obtain the character of any desired irreducible representation in terms of the fermionic fusion-product characters.

Kostka polynomials and their generalizations are polynomials in q, with coefficients which are non-negative integers. The classical Kostka polynomials are known in the theory of symmetric functions [8] as the transition coefficients between Hall–Littlewood and Schur polynomials. In physics, the Kostka polynomials made their first appearance in the study of the completeness of Bethe ansatz states in the Heisenberg spin chain [9, 10]. The methods introduced there, together with the theory of crystal bases, are the basis for the subsequent studies of [7, 6, 11].

In order to establish our notation, in section 2 we provide a brief review of affine Lie algebras. In section 3 we introduce the 'principal subspace' of Feigin and Stoyanovsky, and the function spaces that are dual to them. In section 4 we show how the affine Weyl translations allow us, in the special case of rectangular representations, to use the character of the principal subspace to compute the character of the full space. This is the process that in I we called 'filling the bose sea'. In section 5 we explain why the naïve extension of the method to non-rectangular representations fails, and exhibit the subtractions necessary to obtain correct formulae for the characters. These formulae were originally obtained numerically. In section 6 we show how these correct formulae find their explanation in characters of reducible fusion-product representations, and also explain the origin of the q-dependent Kostka polynomial coefficients in the fusion-product decomposition. In section 7, we make the connection between these Kostka polynomials and the WZW conformal field theory. Finally, in section 8, we combine all the results to obtain a character formula for arbitrary highest weight representations of $\widehat{su}(r+1)_k$, equation (8.2). Section 9 is devoted to the conclusions and an outlook. Details concerning the characters of the principal subspaces can be found in appendix A, while appendix B contains details about the explicit formula for the (generalized) Kostka polynomials.

2. Notation

A simple Lie algebra $\mathfrak{g}$ gives rise to the affine Lie algebra $\widehat{\mathfrak{g}}' = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \widehat{c}\mathbb{C}$ with commutation relations

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{n+m} + \widehat{c}n\langle a, b\rangle \delta_{m+n,0}. \tag{2.1}$$

Here $\langle a, b\rangle$ denotes the Killing form, normalized so that a long root has length $\sqrt{2}$. The element $\widehat{c}$ is central, and so commutes with all $a \in \widehat{\mathfrak{g}}$. It is convenient to adjoin to this algebra a grading operator $\widehat{d}$ that also commutes with $\widehat{c}$, and such that $[\widehat{d}, a \otimes t^n] = n(a \otimes t^n)$. The algebra with the adjoined operator $\widehat{d}$ is denoted by $\widehat{\mathfrak{g}}$. We will often write $a \otimes t^n \equiv a[n]$.

The affine algebra is defined for polynomials in t, t^{-1}. We will later need to extend the definition of functions which are rational functions in t, or more generally, Laurent series in t^{-1}. If $f(t)$ and $g(t)$ are such series, we can write the commutation relations as

$$[a \otimes f, b \otimes g] = [a, b] \otimes fg + \frac{\widehat{c}\langle a, b\rangle}{2\pi \mathrm{i}} \oint f' g \, \mathrm{d}t. \tag{2.2}$$

Here the integral treats the otherwise formal parameter t as a complex variable taking values on a circle surrounding $t = 0$. The result of the integration, $\mathrm{Res}_{t=0}(f'g \, \mathrm{d}t)$, involves only a finite sum because of the polynomial condition on the negative powers of t in f, g.

We will use the Chevalley basis for the generators of the simple algebra $\mathfrak{g}$, in which each simple root α_i is associated with a step-up ladder operator $e_{\alpha_i} \equiv E^{\alpha_i}$, a step-down ladder operator $f_{\alpha_i} \equiv E^{-\alpha_i}$ and the co-root element of the Cartan algebra $h_{\alpha_i} \equiv \frac{2\alpha_i \cdot H}{\|\alpha_i\|^2}$. The commutation relations of these generators are

$$\begin{aligned} \left[h_{\alpha_i}, h_{\alpha_j}\right] &= 0, & \left[h_{\alpha_i}, e_{\alpha_j}\right] &= (\mathbf{C}_r)_{ji} e_{\alpha_j}, \\ \left[h_{\alpha_i}, f_{\alpha_j}\right] &= -(\mathbf{C}_r)_{ji} f_{\alpha_j}, & \left[e_{\alpha_i}, f_{\alpha_j}\right] &= \delta_{ij} h_{\alpha_i}. \end{aligned} \tag{2.3}$$

The remaining generators are obtained by repeated commutators of these, subject to the *Serre relations*

$$\left[\mathrm{ad}\big(e_{\alpha_i}\big)\right]^{1-(\mathbf{C}_r)_{ji}} e_{\alpha_j} = 0, \qquad \left[\mathrm{ad}\big(f_{\alpha_i}\big)\right]^{1-(\mathbf{C}_r)_{ji}} f_{\alpha_j} = 0. \tag{2.4}$$

In these expressions $(\mathbf{C}_r)_{ij}$ denotes the elements of the Cartan matrix of $\mathfrak{g}$:

$$(\mathbf{C}_r)_{ij} \stackrel{\mathrm{def}}{=} \frac{2(\alpha_i \cdot \alpha_j)}{\|\alpha_j\|^2}. \tag{2.5}$$

In the case $\mathfrak{g} = su(r+1)$ the elements of the Cartan matrix are given by $(\mathbf{C}_r)_{ij} = 2\delta_{i,j} - \delta_{|i-j|,1}$. We will also have cause to use the inverse Cartan matrix whose elements (for $\mathfrak{g} = su(r+1)$) are $\left(\mathbf{C}_r^{-1}\right)_{ij} = \min(i,j) - \frac{ij}{r+1}$.

Our interest is in *integrable* representations of $\widehat{\mathfrak{g}}$, and in particular of $\widehat{su}(r+1)$. An integrable representation is one that, under restriction to any subgroup of $\widehat{\mathfrak{g}}$ isomorphic to $su(2)$, decomposes into a set of finite-dimensional representations of this $su(2)$. From the work of Kac [12] it is known that in any irreducible integrable representation the generator $\widehat{c}$ will act as a positive integer multiple of the identity, $\widehat{c} \mapsto k\mathbb{I}$, and we will follow the physics convention of appending the integer k, the *level* of the representation, to the name of the group, and so write $\widehat{\mathfrak{g}}_k$. These integrable representations are all highest weight representations whose highest weight vector is annihilated by $e_{\alpha_i}[n]$, $n \geqslant 0$, and by $h_{\alpha_i}[n]$, $f_{\alpha_i}[n]$ with $n > 0$. The top d-graded subspaces (where $\widehat{d}$ is taken to act as zero) form finite-dimensional representations of the simple algebra $\mathfrak{g}$, but not all representations of $\mathfrak{g}$ can form top components of integrable representations of $\widehat{\mathfrak{g}}_k$. In the case of $\widehat{su}(r+1)_k$ the restriction on the representations of $su(r+1)$ that can appear as top components is that the number of columns in the Young diagram labelling the representation must be $\leqslant k$.

We wish to obtain the dimensions $\mathrm{mult}(\widehat{\mu})$ of the weight spaces $\widehat{\mu} \equiv (\mu; k; d)$ in an irreducible integrable level-k representation $H_{\widehat{\lambda}}$ of $\widehat{\mathfrak{g}}_k$, whose highest weight is $\widehat{\lambda} = (\lambda; k; 0)$. Here λ and μ denote a highest weight and an arbitrary weight, respectively, of the associated finite-dimensional simple algebra $\mathfrak{g}$, and d, a non-positive integer, is the eigenvalue of the grading operator $\widehat{d}$. The character of the representation is then defined to be

$$\mathrm{ch}\, H_{\widehat{\lambda}}(q, \mathbf{x}) = \sum_{\widehat{\mu}} \mathrm{mult}(\widehat{\mu}) x_1^{\mu_1} x_2^{\mu_2} \cdots x_r^{\mu_r} q^{-d}. \tag{2.6}$$

Here $\mu_1, \ldots, \mu_r$ are the components of the weight μ in the basis of the fundamental weights of the finite-dimensional algebra $\omega_i = \sum_j \left(\mathbf{C}_r^{-1}\right)_{ji} \alpha_j$, i.e. $\mu = \sum_{i=1}^r \mu_i \omega_i$, and

$\mathbf{x} = (x_1, \ldots, x_r)$. Because the eigenvalues d are zero or negative, the character is a formal series in positive powers of q.

In this paper, we present fermionic formulae for the characters of all highest weight integrable representations of $\widehat{su}(r+1)$. These formulae are inspired by the construction of Feigin and Stoyanovsky, which we present below.

3. The principal subspace

In this section we briefly review the strategy of Feigin and Stoyanovsky [2], as we presented it in I. The reader should refer to I for more details.

Let $|\widehat{\lambda}\rangle$ be the highest weight state of the irreducible representation $H_{\widehat{\lambda}}$. Recall that e_{α_i} and f_{α_i} are the step-up and step-down ladder operators corresponding to simple root α_i in the finite-dimensional algebra $\mathfrak{g}$, and $e_{\alpha_i}[n]$ and $f_{\alpha_i}[n]$ are their $\widehat{\mathfrak{g}}$ relatives. Since $H_{\widehat{\lambda}}$ is irreducible, by acting on $|\widehat{\lambda}\rangle$ with products of $e_{\alpha_i}[n], n < 0$, and $f_{\alpha_i}[n], n \leqslant 0$, we generate the entire space $H_{\widehat{\lambda}}$. If we restrict ourselves to products containing only the $f_{\alpha_i}[n], n \leqslant 0$, we obtain what is known as the *principal subspace* $W_{\widehat{\lambda}}$.

Note that in I we used a commuting set of operators applied to the highest weight state to generate a principal subspace (in the case of $su(2)$ and $su(3)$). This subspace had the advantage for physics applications that the resulting correlation functions were polynomials, rather than rational functions. For the purposes of this paper, however, it is necessary to use the Feigin–Stoyanovsky subspace, in order to obtain the simple closed-form formulae for the principal subspace multiplicities for arbitrary representations that we report here.

Consider, for example, $su(2)$ where there is only one simple root α_1 and so the root label is redundant. Here, since each application of an $f[-n]$ moves us one step to the left and n steps down in the weight diagram, we cannot reach any weight to the right of the column lying below λ. (See figure 2 where the upper part of the weight diagram of the vacuum representation of $\widehat{su}(2)_2$ is displayed.) Furthermore, we can reach weights close to this column via fewer distinct products $f[-n_1]f[-n_2]\cdots f[-n_m]$ than the dimension of the weight space. Because of this paucity of paths, even if all the $f[-n_1]f[-n_2]\cdots f[-n_m]|\widehat{\lambda}\rangle$ were linearly independent, we would be able to obtain only a subspace in each weight space. If, however, we look at weights in columns far to the left in the weight diagram, the number of distinct paths leading to a given weight grows rapidly, whilst the dimensions of the weight spaces near the head of any such column remain small. It is plausible, and indeed true, that we can obtain all states in such a weight space by applying suitable products of $f[-n]$'s to $|\widehat{\lambda}\rangle$. Thus, by counting the number of linearly independent paths, we can obtain the multiplicity of these distant weights. Because the weight diagram is invariant under the action of affine Weyl translations, as we explain in section 4, this information provides the dimension of *all* the weight spaces, and hence the complete character.

We could in principle use Lie algebra relations to determine the number of independent paths between the weights. It is easier, however, to obtain the dimensions of the weights $\widehat{\mu}$ appearing in the principal subspace by counting the number of linearly independent functions F that arise as matrix elements

$$F(\{z^{(1)}\}, \{z^{(2)}\}, \ldots) = \langle v|R\big\{f_{\alpha_1}\big(z_1^{(1)}\big)\cdots f_{\alpha_1}\big(z_{m^{(1)}}^{(1)}\big)f_{\alpha_2}\big(z_1^{(2)}\big)\cdots f_{\alpha_2}\big(z_{m^{(2)}}^{(2)}\big)\cdots\big\}|\widehat{\lambda}\rangle, \tag{3.1}$$

where $|v\rangle$ is an element of a weight space $\widehat{\mu}$.

In addition, $f_{\alpha_i}(z)$ denotes the current operator

$$f_{\alpha_i}(z) = \sum_{n=-\infty}^{\infty} f_{\alpha_i}[n] z^{-n-1}, \tag{3.2}$$

and R denotes 'radial ordering', meaning that operators $f(z)$ with larger $|z|$ are placed to the left of those with smaller $|z|$'s[1]. By duality, the number of linearly independent functions F with $m^{(i)}$ currents $f_{\alpha_i}(z)$ and the fixed total degree N is equal to the dimension, in the principal subspace $W_{\widehat{\lambda}}$ of the weight

$$\widehat{\mu} = \left(\lambda - \sum_i m^{(i)}\alpha_i; k; -N - \sum_i m^{(i)}\right). \tag{3.3}$$

Thus, the numbers $m^{(i)}$ correspond to the number of roots α_i which need to be subtracted from λ to obtain μ, hence, in terms of the fundamental weights, we have $\mu = \sum_{i=1}^{r}\left(l_i - \sum_j (\mathbf{C}_r)_{ij} m^{(j)}\right)\omega_i$, where the l_i are the components of λ, i.e. $\lambda = \sum_i l_i\omega_i$.

The linear dependencies between products of $f_{\alpha_i}[n]$ due to relations in the Lie algebra, the integrability, and the properties of the highest weight vector $|\widehat{\lambda}\rangle$ translate, through duality, into conditions satisfied by the function F. For example, operators $f_{\alpha_i}(z)$ corresponding to the same simple root commute with each other. The function F is therefore a symmetric function in the $m^{(i)}$ variables $z_j^{(i)}$ for fixed i. It must also possess certain poles and zeros whose exact form we will specify below. Once we know all these properties, we can set out to count the number of independent functions. It is not, however, easy to be sure that we have obtained a complete set of constraints on F. If we have missed some, we will over-count the multiplicities.

In the case of $\widehat{su}(r+1)_k$ representations with only one non-zero Dynkin index, i.e. $\lambda = l\omega_p$, it is not too hard to find all the constraints on functions $F\left(\left\{z_j^{(i)}\right\}\right)$. We already observed that $F\left(\left\{z_j^{(i)}\right\}\right)$ is symmetric under the exchange of the variables $z_j^{(i)} \leftrightarrow z_{j'}^{(i)}$. Next we observe that the commutator

$$\left[f_{\alpha_i}, f_{\alpha_{i+1}}\right] = f_{\alpha_i+\alpha_{i+1}} \tag{3.4}$$

implies the operator product

$$f_{\alpha_i}(z) f_{\alpha_{i+1}}(w) = \frac{f_{\alpha_i+\alpha_{i+1}}(w)}{(z-w)} + \text{regular terms}, \tag{3.5}$$

and this in turn implies that $F\left(\left\{z_j^{(i)}\right\}\right)$ can have a pole of at most order one when the coordinates of two currents corresponding to adjacent simple roots coincide, $z_j^{(i)} = z_{j'}^{(i+1)}$. (There is no pole for non-adjacent indices because $\alpha_i + \alpha_j$ is not a root unless $j = i \pm 1$.) We will refer to the index i as the *colour* of the variables. In the representation $\lambda = l\omega_p$, and for $l > 0$, we have that $f_{\alpha_p}[0]|\widehat{\lambda}\rangle \neq 0$, and since $f_{\alpha_p}[0]$ comes with coefficient z^{-1}, the function $F\left(\left\{z_j^{(i)}\right\}\right)$ may have a pole of order one at $z_j^{(p)} = 0$. (Note that $f_{\alpha_i}[0]|\widehat{\lambda}\rangle = 0$ for $i \neq p$.) These are the only possible poles, and we make them explicit by writing

$$F\left(\left\{z_j^{(i)}\right\}\right) = \frac{f\left(\left\{z_j^{(i)}\right\}\right)}{\prod_j \left(z_j^{(p)}\right) \prod_{i=1}^{r-1}\prod_{j,j'}\left(z_j^{(i)} - z_{j'}^{(i+1)}\right)}, \tag{3.6}$$

where $f\left(\left\{z_j^{(i)}\right\}\right)$ is now a *polynomial* symmetric under the exchange of variables of the same colour: $z_j^{(i)} \leftrightarrow z_{j'}^{(i)}$. Because of relations in the algebra and properties of the representation $f\left(\left\{z_j^{(i)}\right\}\right)$ is not an arbitrary symmetric polynomial, but must possess certain zeros. We now describe these as follows:

(i) The integrability condition requires that $\left[f_{\alpha_i}(z)\right]^{k+1}$ annihilates *any* vector in the representation. This tells us that $f\left(\left\{z_j^{(i)}\right\}\right) = 0$ when $z_1^{(i)} = z_2^{(i)} = \cdots = z_{k+1}^{(i)}$, for any colour i.

[1] Strictly speaking, we regard $z_i^{(\alpha)}$'s as formal variables and consider matrix elements of all possible orderings of the roots α_i. We then count the resulting number of linearly independent functions in the formal variables $z_i^{(\alpha)}$.

(ii) The integrability properties of the top component of the representation with highest weight $\lambda = l\omega_p$ tell us that $f_{\alpha_p}^{l+1}[0]|\widehat{\lambda}\rangle = 0$. The function $f\left(\left\{z_j^{(i)}\right\}\right)$ must therefore have a zero when $l+1$ of the $z_j^{(p)}$ becomes zero.

(iii) The Serre relations (2.4) are

$$\left[f_{\alpha_i}, \left[f_{\alpha_i}, f_{\alpha_{i+1}}\right]\right] = \left[f_{\alpha_{i+1}}, \left[f_{\alpha_{i+1}}, f_{\alpha_i}\right]\right] = 0, \tag{3.7}$$

and indicate that F should have no pole if two currents of colour i are made to coincide with a current of adjacent colour. This requires that $f\left(\left\{z_j^{(i)}\right\}\right)$ has a pole-cancelling zero when

$$z_1^{(i)} = z_2^{(i)} = z_1^{(i+1)} \qquad \text{or when} \quad z_1^{(i)} = z_1^{(i+1)} = z_2^{(i+1)}, \qquad \text{for} \quad i = 1, \ldots, r-1.$$

To summarize: we find that the polynomial $f\left(\left\{z_j^{(i)}\right\}\right)$ is symmetric in the variables corresponding to the same colour, and vanishes when any of the following conditions holds:

$$z_1^{(i)} = \cdots = z_{k+1}^{(i)}, \quad \forall i \tag{3.8}$$

$$z_1^{(i)} = z_2^{(i)} = z_1^{(i+1)}, \qquad z_1^{(i)} = z_1^{(i+1)} = z_2^{(i+1)}, \quad i = 1, \ldots, r-1, \tag{3.9}$$

$$z_1^{(p)} = \cdots = z_{l+1}^{(p)} = 0. \tag{3.10}$$

We will denote the space of rational functions $F\left(\left\{z_j^{(i)}\right\}\right)$ (3.6) such that $f\left(\left\{z_j^{(i)}\right\}\right)$ satisfies (3.8), (3.9) and (3.10) by $\mathcal{F}_{l\omega_p;k}$. We now define the character of this space to be

$$\mathrm{ch}\, \mathcal{F}_{l\omega_p;k}(q, \mathbf{x}) \stackrel{\mathrm{def}}{=} x_p^l \sum_{\{F(z)\}} q^{\deg(F) + \sum_{i=1}^r m^{(i)}} \left(\prod_{i=1}^r x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}} \right), \tag{3.11}$$

where the sum is over all the functions $F\left(\left\{z_j^{(i)}\right\}\right)$ in the space $\mathcal{F}_{l\omega_p;k}$. The powers of q and x_i are motivated by the form of the weights (3.3), which we rewrite here in terms of the fundamental weights ω_i

$$\widehat{\mu} = \left(\lambda - \sum_{i,j} m^{(j)} (\mathbf{C}_r)_{ji} \omega_i; k; -N - \sum_i m^{(i)} \right). \tag{3.12}$$

As a reminder, $m^{(i)}$ is the number of variables of colour (i), which corresponds to the number of roots α_i subtracted from the highest weight $\lambda = l\omega_p$. Note that the exponent of q is not just the total degree of $F(z)$, but is shifted by $\sum_i m^{(i)}$ (compare with (3.3)). The reason is that the currents $f_{\alpha_i}(z)$ are defined by equation (3.2), in which the power of z is shifted by the scaling dimension.

Counting the dimensions of these function spaces is an intricate but tractable problem. We will explain the structure of this character in the appendix A, and refer to [5] for details and the proof. The result is the function-space character

$$\mathrm{ch}\, \mathcal{F}_{l\omega_p;k}(q, \mathbf{x}) = x_p^l \sum_{\substack{i=1,\ldots,r \\ a=1,\ldots,k \\ m_a^{(i)} \geqslant 0}} \left(\prod_{i=1}^r x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}} \right) \frac{q^{\frac{1}{2} m_a^{(i)} (\mathbf{C}_r)_{i,j} \mathbf{A}_{a,b} m_b^{(j)} - \mathbf{A}_{a,l} m_a^{(p)}}}{\prod_{i=1}^r \prod_{a=1}^k (q)_{m_a^{(i)}}}, \tag{3.13}$$

where it is understood that repeated indices are summed over. Here various symbols need to be defined: (i) the matrix $\mathbf{A}$ has the entries $\mathbf{A}_{a,b} = \min(a, b)$; (ii) for any integer $m > 0$, we define $(q)_m = \prod_{i=1}^m (1 - q^i)$ and $(q)_0 = 1$; (iii) the sum over integers $m_a^{(i)}$ is to be understood as a sum over partitions of $m^{(i)}$, where $m_a^{(i)}$ denotes the number of rows of length a in partition (i). That is $\sum_{a=1}^k a m_a^{(i)} = m^{(i)}$, see figure 1 for an example.

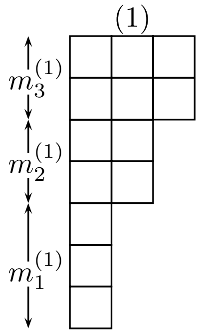


Figure 1. A partition of $m^{(1)} = 13$.

The space of functions $\mathcal{F}_{l\omega_p;k}$ is dual to the principal subspace $W_{l\omega_p;k}$. It follows that the character (3.13) is the character of the principal subspace

$$\operatorname{ch} W_{l\omega_p;k}(q,\mathbf{x}) = \operatorname{ch} \mathcal{F}_{l\omega_p;k}(q,\mathbf{x}). \tag{3.14}$$

In the next section, we will explain how we can use the affine Weyl translations to obtain characters for the full representation from the characters of the principal subspaces.

We would like to note that it is straightforward to generalize the function space $\mathcal{F}_{l\omega_p;k}$, by changing the constraint (3.10) to $z_1^{(i)} = \cdots = z_{l_i+1}^{(i)} = 0$ for $i = 1,\dots,r$ and taking the product over all p in the denominator of (3.6). We will denote this function space by $\mathcal{F}_{\mathbf{l};k}$, where $\mathbf{l} = (l_1,\dots,l_r)^T$. The character of this function space is

$$\operatorname{ch} \mathcal{F}_{\mathbf{l};k}(q,\mathbf{x}) = \prod_{i=1}^{r} x_i^{l_i} \sum_{\substack{i=1,\dots,r\\ a=1,\dots,k\\ m_a^{(i)}\geqslant 0}} \left(\prod_{i=1}^{r} x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}}\right) \frac{q^{\frac{1}{2} m_a^{(i)} (\mathbf{C}_r)_{i,j} \mathbf{A}_{a,b} m_b^{(j)} - \mathbf{A}_{a,l_i} m_a^{(i)}}}{\prod_{i=1}^{r} \prod_{a=1}^{k} (q)_{m_a^{(i)}}}. \tag{3.15}$$

We would like to stress that this character is not the character of the principal subspace $W_{\lambda;k}$, where $\lambda = \sum_i l_i\omega_i$. However, it can be interpreted as a character of a somewhat larger space, as we will see below.

4. Affine Weyl translations

In this section, we will exploit the symmetry of the representation spaces under the affine Weyl group. Elements of the affine Weyl group map weights of a highest weight representation to other weights in the same representation, in such a way that the weight-space multiplicities are preserved.

Elements of the affine Weyl group can be thought of as a product of a finite Weyl reflection and an affine Weyl translation. We will focus on the abelian subgroup generated by the affine Weyl translations, because they will enable us to obtain the character of the full integrable representation from the character of the principal subspace.

The affine Weyl translation $T_{\alpha_i}^{N_i}$ acts on a weight $\hat{\lambda} = (\lambda; k; d)$, where $\lambda = \sum_i l_i\omega_i$, by 'translating' it to $\lambda + N_i\alpha_i$ (no summation implied) and shifting the value of d (see, for instance, [12], equation (6.5.2))

$$T_{\alpha_i}^{N_i}(\lambda; k; d) = \big(\lambda + kN_i\alpha_i; k; d - N_i l_i - N_i^2 k\big). \tag{4.1}$$

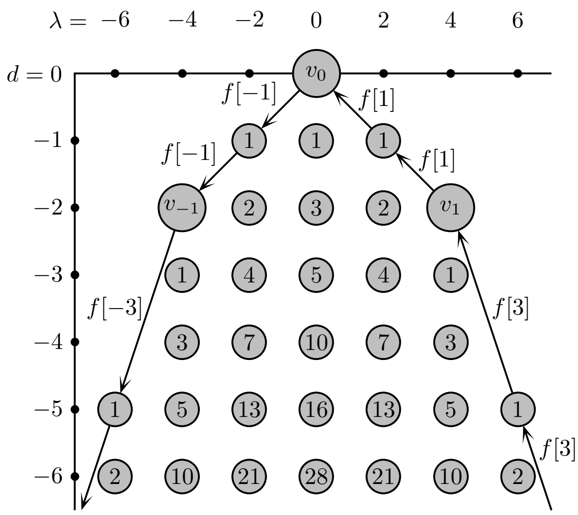


Figure 2. The top part of the weight diagram of the vacuum representation of $\widehat{su}(2)_2$. The numbers denoted the dimension of the corresponding weight space.

More generally, we have

$$\prod_{i=1}^{r} T_{\alpha_i}^{N_i}(\lambda; k; d) = \left(\lambda + k\sum_{i=1}^{r} N_i\alpha_i; k; d - \sum_{i=1}^{r} N_i l_i - \frac{k}{2}\mathbf{N}^T \cdot \mathbf{C}_r \cdot \mathbf{N}\right), \tag{4.2}$$

where $\mathbf{N} = (N_1, \ldots, N_r)^T$, and $\mathbf{C}_r$ is the Cartan matrix of $su(r+1)$.

We can use these affine Weyl translations to obtain the characters of the full representation. To illustrate how this works, we will use an explicit example, namely the vacuum representation of $\widehat{su}(2)_2$. The top part of the weight diagram of this representation is given in figure 2.

From this figure, we see that we can obtain the whole representation by acting with an arbitrary combination of the operators $e[i]$ and $f[i]$, with $i < 0$ on the highest weight state $|\hat{\lambda}\rangle \equiv |v_0\rangle$ (we dropped the subscript α_1). Under the affine Weyl translation T^1, this weight is mapped to the weight $|v_1\rangle$. We can also generate the whole representation by acting with operators $e[i]$ and $f[j]$ on this state. In this case, we need to act with an arbitrary combination of operators taken from the sets $\{e[i]|i < -2\}$ and $\{f[j]|j < 2\}$. In general, the whole representation can be obtained by acting on the state $|v_N\rangle = T^N|v_0\rangle$ with the operators $\{e[i]|i < -2N\}$ and $\{f[j]|j < 2N\}$. We can now take the limit $N \to \infty$, and obtain that, we can generate the whole representation by acting on $|v_\infty\rangle$ with only the step-down operators $\{f[j]|j \in \mathbb{Z}\}$, see [13, 14] for the proof of this statement.

Let us come back to the principal subspace. As a reminder, the principal subspace $W_{\hat{\lambda}}$ is the space generated by acting with the operators $f[j]$, with $j < 0$ on $|v_0\rangle$ (again, we are focusing on the vacuum representation of $\widehat{su}(2)_2$). We can define a sequence of subspaces, by acting with the affine Weyl translation, i.e. $W^{(N)} = T^N W_{\hat{\lambda}}$. The subspace $W^{(N)}$ is obtained by acting with the operators $\{f[j]|j < -2N\}$ on the state $|v_N\rangle$. So, in the limit $N \to \infty$, we find that $W^{(\infty)}$ is obtained by acting with $\{f[j]|j \in \mathbb{Z}\}$, on the state $|v_\infty\rangle$. Comparing this subspace with the description of the full representation of the previous paragraph, we find that they are in fact the same.

Using this result, we can obtain the characters of the integrable representation by acting with the affine Weyl translation T^N on the character of the principal subspace, and taking the limit $N \to \infty$.

In our paper [1], we showed that the effect of acting with the affine Weyl translation and taking the limit of $N \to \infty$ only alters the character of the principal subspaces of $\widehat{su}(2)_k$ in the following two ways. First of all, the summation over the variable $m_k^{(1)} \geqslant 0$ is extended to the negative integers. Secondly, the factor $\frac{1}{(q)_{m_k^{(1)}}}$ is replaced by $\frac{1}{(q)_\infty}$. In physical terms (see [1]), this corresponds to 'filling the Bose sea', and considering the excitations on top of a 'large droplet'. In [5], we showed that this procedure also works in the case $\widehat{su}(r+1)_k$. Using this result, we obtain the characters for rectangular highest weight representations

$$\mathrm{ch}\, H_{l\omega_p;k}(q,\mathbf{x}) = \frac{1}{(q)_\infty^r} x_p^l \sum_{\substack{m_k^{(i)} \in \mathbb{Z} \\ m_{a<k}^{(i)} \in \mathbb{Z}_{\geqslant 0} \\ i=1,\dots,r}} \left(\prod_{i=1}^r x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}} \right) \frac{q^{\frac{1}{2} m_a^{(i)} (\mathbf{C}_r)_{i,j} \mathbf{A}_{a,b} m_b^{(j)} - \mathbf{A}_{a,l} m_a^{(p)}}}{\prod_{i=1}^r \prod_{a=1}^{k-1} (q)_{m_a^{(i)}}}. \tag{4.3}$$

The character (4.3) can be written in a form which allows all l_i to be non-zero, namely, in the same way as the character $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}$, equation (3.15)

$$\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty(q,\mathbf{x}) \stackrel{\mathrm{def}}{=} \frac{1}{(q)_\infty^r} \left(\prod_{i=1}^r x_i^{l_i} \right) \sum_{\substack{m_k^{(i)} \in \mathbb{Z} \\ m_{a<k}^{(i)} \in \mathbb{Z}_{\geqslant 0} \\ i=1,\dots,r}} \left(\prod_{i=1}^r x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}} \right) \frac{q^{\frac{1}{2} m_a^{(i)} (\mathbf{C}_r)_{i,j} \mathbf{A}_{a,b} m_b^{(j)} - \mathbf{A}_{a,l_i} m_a^{(i)}}}{\prod_{i=1}^r \prod_{a=1}^{k-1} (q)_{m_a^{(i)}}}. \tag{4.4}$$

We cannot however interpret this character as the character of the highest weight representation $H_{\hat{\lambda}}$, where the finite part of $\hat{\lambda}$ is given by $\lambda = \sum_i l_i \omega_i$. This is because we did not take all the constraints into account for arbitrary representations. For example, for general highest weight, our restrictions on the space do not ensure that $f_\alpha[1]|\widehat{\lambda}\rangle = 0$ whenever α is not a simple positive root. However, as we will show in the next section, the characters $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty$ are an essential ingredient of the characters $\mathrm{ch}\, H_{\hat{\lambda};k}$ for arbitrary highest weights.

5. Arbitrary highest weight representations

As mentioned above, we do not expect the function-space characters $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty$ defined in equation (4.4) to be the characters of arbitrary highest weight representations $H_{\hat{\lambda}}$. This is because we have not yet imposed all the conditions on the function spaces. We have not imposed the remaining conditions, because the complexity of the resulting space makes counting its dimensions intractable.

We anticipate, therefore, that the function-space character (4.4) over-counts the dimensions of the weight spaces whenever λ is not a rectangular representation. Nevertheless it is reasonable to conjecture some sort of relation between $\mathrm{ch}\, H_{\hat{\lambda}}$ and $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty$.

To seek such a relation we used *Mathematica*® to compare the multiplicities given by $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty$ with those in the representation $H_{\hat{\lambda}}$, obtained by using the affine version of Freudenthal's recursion formula (see for instance, [15], page 578). We found that whenever more than one Dynkin index is non-zero, the character $\mathrm{ch}\, \mathcal{F}_{\mathbf{l};k}^\infty$ does indeed over-count, and subtractions are necessary. It turns out that these subtractions can be written as a sum over $\mathrm{ch}\, \mathcal{F}_{\mathbf{l}';k}^\infty$, where the coefficients are polynomials in q^{-1}.

To illustrate this we give a table expressing the characters of integrable level-4 representations of $\widehat{su}(4)$ in terms of the characters $\mathrm{ch}\, \mathcal{F}_{\mathbf{l}';k}^\infty$.

$$\begin{aligned}
\mathrm{ch}\, H_{1,1,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,1;4} \\
\mathrm{ch}\, H_{2,1,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{2,1,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,0,1;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,0;4} \\
\mathrm{ch}\, H_{1,2,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{1,2,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,1,1;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,0,0;4} \\
\mathrm{ch}\, H_{1,1,1;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,1;4} - \left(\frac{1}{q}+\frac{1}{q^2}\right)\mathrm{ch}\,\mathcal{F}^{\infty}_{0,1,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{2,0,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,2;4} \\
\mathrm{ch}\, H_{3,1,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{3,1,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{2,0,1;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,0,0;4} \\
\mathrm{ch}\, H_{2,2,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{2,2,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,1;4} + \left(\frac{1}{q^2}+\frac{1}{q^3}\right)\mathrm{ch}\,\mathcal{F}^{\infty}_{0,1,0;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{2,0,0;4} \\
\mathrm{ch}\, H_{2,1,1;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{2,1,1;4} - \left(\frac{1}{q}+\frac{1}{q^2}\right)\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{3,0,0;4} \\
&\quad - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,0,2;4} + \left(\frac{1}{q^2}+\frac{1}{q^3}\right)\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,1;4} \\
\mathrm{ch}\, H_{1,3,0;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{1,3,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,2,1;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,0;4} - \frac{1}{q^3}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,1;4} \\
\mathrm{ch}\, H_{1,2,1;4} &= \mathrm{ch}\,\mathcal{F}^{\infty}_{1,2,1;4} - \left(\frac{1}{q}+\frac{1}{q^2}\right)\mathrm{ch}\,\mathcal{F}^{\infty}_{0,2,0;4} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{2,1,0;4} \\
&\quad - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,1,2;4} + \frac{1}{q^2}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,0,1;4} - \frac{1}{q^3}\mathrm{ch}\,\mathcal{F}^{\infty}_{0,0,0;4}.
\end{aligned} \tag{5.1}$$

In addition, we found evidence for the relation

$$\mathrm{ch}\, H_{l_1,0,l_3;k} = \mathrm{ch}\,\mathcal{F}^{\infty}_{l_1,0,l_3;k} - \frac{1}{q}\mathrm{ch}\,\mathcal{F}^{\infty}_{l_1-1,0,l_3-1;k}, \tag{5.2}$$

for $l_1, l_3 > 0$ and $k \leqslant 4$.

It is interesting that the characters $\mathrm{ch}\, H_{\lambda;k}$ can still be expressed in terms of $\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{l};k}$, but the pattern of subtractions is at first sight obscure. However, inverting the relations so as to express the characters $\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{l};k}$ in terms of the characters of the full representations gives a clue as to what is happening. We find

$$\begin{aligned}
\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,0;4} &= \mathrm{ch}\, H_{1,1,0;4} + \frac{1}{q}\mathrm{ch}\, H_{0,0,1;4} \\
\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,1;4} &= \mathrm{ch}\, H_{1,1,1;4} + \left(\frac{1}{q}+\frac{1}{q^2}\right)\mathrm{ch}\, H_{0,1,0;4} + \frac{1}{q}\mathrm{ch}\, H_{2,0,0;4} + \frac{1}{q}\mathrm{ch}\, H_{0,0,2;4} \\
\mathrm{ch}\,\mathcal{F}^{\infty}_{2,1,0;4} &= \mathrm{ch}\, H_{2,1,0;4} + \frac{1}{q}\mathrm{ch}\, H_{1,0,1;4} \\
\mathrm{ch}\,\mathcal{F}^{\infty}_{1,2,0;4} &= \mathrm{ch}\, H_{1,2,0;4} + \frac{1}{q}\mathrm{ch}\, H_{0,1,1;4} \\
\mathrm{ch}\,\mathcal{F}^{\infty}_{3,1,0;4} &= \mathrm{ch}\, H_{3,1,0;4} + \frac{1}{q}\mathrm{ch}\, H_{2,0,1;4} \\
\mathrm{ch}\,\mathcal{F}^{\infty}_{1,3,0;4} &= \mathrm{ch}\, H_{1,3,0;4} + \frac{1}{q}\mathrm{ch}\, H_{0,2,1;4}
\end{aligned}$$

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{2,2,0;4} = \mathrm{ch}\,H_{2,2,0;4} + \frac{1}{q}\mathrm{ch}\,H_{1,1,1;4} + \frac{1}{q^2}\mathrm{ch}\,H_{0,0,2;4}$$

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{2,1,1;4} = \mathrm{ch}\,H_{2,1,1;4} + \left(\frac{1}{q} + \frac{1}{q^2}\right)\mathrm{ch}\,H_{1,1,0;4} + \frac{1}{q}\mathrm{ch}\,H_{3,0,0;4} + \frac{1}{q}\mathrm{ch}\,H_{1,0,2;4} + \frac{1}{q^2}\mathrm{ch}\,H_{0,0,1;4}$$

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{1,2,1;4} = \mathrm{ch}\,H_{1,2,1;4} + \left(\frac{1}{q} + \frac{1}{q^2}\right)\mathrm{ch}\,H_{0,2,0;4} + \frac{1}{q}\mathrm{ch}\,H_{2,1,0;4} + \frac{1}{q}\mathrm{ch}\,H_{0,1,2;4} + \frac{1}{q^2}\mathrm{ch}\,H_{1,0,1;4}. \tag{5.3}$$

In addition, we have

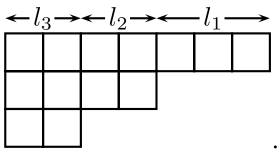
$$\mathrm{ch}\,\mathcal{F}^{\infty}_{l_1,0,l_3;k} = \sum_{j=0}^{\min(l_1,l_3)} \frac{1}{q^j}\mathrm{ch}\,H_{l_1-j,0,l_3-j;k}. \tag{5.4}$$

All the signs on the right-hand side of the above decompositions are positive. This suggests that the $\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{l};k}$ are indeed characters of $\widehat{su}(r+1)$ representations, but these representations are *reducible*. We need to understand which representations are occurring, and why the coefficients of their characters are polynomials in q^{-1}. This we will do in the next section.

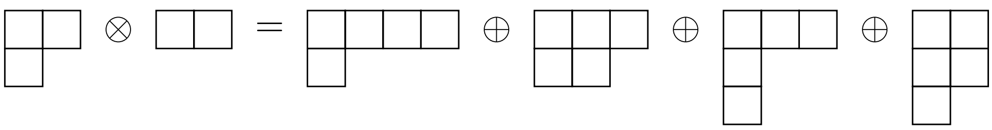
6. Kostka polynomials

In this section we will introduce Feigin and Loktev's q-refinement [16] of the Littlewood–Richardson coefficients that occur in the decomposition of tensor products of the finite-dimensional $su(r+1)$ representations [17].

Let ω_i denote the fundamental weights of the finite-dimensional $su(r+1)$ algebra. A finite-dimensional irreducible representation labelled by the Dynkin indices $l_i \in \mathbb{Z}_{\geqslant 0}$ has the highest weight $\lambda = \sum_{i=1}^{r} l_i\omega_i$ and corresponds to a Young diagram with l_1 columns containing one box, l_2 columns with two boxes, and so on.



The Littlewood–Richardson rules [17] provide a product that allows us to make the space of Young diagrams into an associative algebra. The multiplication operation in this algebra mirrors the multiplication and decomposition of the Schur functions corresponding to the Young diagrams, and, these being the characters of the associated $su(r+1)$ representations, reflect the decomposition of $su(r+1)$ tensor product representations into their irreducible components. For example, using the notation V_{l_1,l_2,l_3} for the $su(4)$ representation with Dynkin indices l_1, l_2, l_3, the Young diagram manipulation



corresponds to the decomposition

$$V_{1,1,0} \otimes V_{2,0,0} = V_{3,1,0} \oplus V_{1,2,0} \oplus V_{2,0,1} \oplus V_{0,1,1}. \tag{6.1}$$

The coefficients are not always unity. For example

$$\ydiagram{1,1,1} \otimes \ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{3,2,1} \oplus \ydiagram{2,2,2} \oplus \ydiagram{2} \oplus 2\,\ydiagram{1,1},$$

or

$$V_{0,0,1} \otimes V_{0,1,0} \otimes V_{1,0,0} = V_{1,1,1} \oplus V_{0,0,2} \oplus V_{2,0,0} \oplus 2V_{0,1,0}. \tag{6.2}$$

This last expression is to be compared with the decomposition of our $\widehat{su}(4)_{k=4}$ function space

$$\operatorname{ch} \mathcal{F}^{\infty}_{1,1,1;4} = \operatorname{ch} H_{1,1,1;4} + \frac{1}{q}\operatorname{ch} H_{0,0,2;4} + \frac{1}{q}\operatorname{ch} H_{2,0,0;4} + \left(\frac{1}{q} + \frac{1}{q^2}\right) \operatorname{ch} H_{0,1,0;4}. \tag{6.3}$$

If we set $q = 1$ in the coefficients in this decomposition, we recover the integers appearing in (6.2). All representations appearing in (5.3) are similarly accounted for: the $H_{\widehat{\lambda}_i}$ appearing in the decomposition of $\operatorname{ch} \mathcal{F}^{\infty}_{l_1,l_2,l_3;4}$ are precisely the $\widehat{su}(4)$ representations whose top grades are the V_{λ_i} in the decomposition of the product $V_{l_1,0,0} \otimes V_{0,l_2,0} \otimes V_{0,0,l_3}$, and if $q \to 1$ their q-coefficients reduce to the multiplicity of these V_{λ_i}.

We need to understand why the coefficients are q-dependent. Following Feigin and Loktev [16], we now show that these q-polynomial coefficients can be introduced even for finite-dimensional $su(r+1)$ representations. Let V_{λ_i} denote a collection of irreducible finite-dimensional highest weight representations of a Lie algebra $\mathfrak{g}$, and consider the tensor product

$$V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}. \tag{6.4}$$

To each V_{λ_i} we associate a distinct complex number ζ_i (to be thought of as 'where' the representation is located). We then introduce the algebra $\mathfrak{g}[t] \equiv \mathfrak{g} \otimes \mathbb{C}[t]$ of $\mathfrak{g}$-valued polynomials in t, with Lie bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n}. \tag{6.5}$$

(Since only non-negative powers of t are being allowed, there is no non-trivial central extension, so this is *not* the affine Lie algebra associated with $\mathfrak{g}$.)

The representation V_{λ_i} gives rise to the *evaluation representation* of $\mathfrak{g}[t]$ by setting

$$(a \otimes t^m)|u\rangle = \zeta_i^m a|u\rangle, \qquad \text{for} \quad |u\rangle \in V_{\lambda_i}. \tag{6.6}$$

We similarly define the action of $\mathfrak{g}$ on the tensor product space. If $a \in \mathfrak{g}$, and $|u_i\rangle \in V_{\lambda_i}$, we set

$$(a \otimes t^m)(|u_1\rangle \otimes \cdots \otimes |u_N\rangle) = \sum_i \zeta_i^m (|u_1\rangle \otimes \cdots \otimes a|u_i\rangle \otimes \cdots \otimes |u_N\rangle). \tag{6.7}$$

We will call this action the geometric, or fusion, co-product. For $m = 0$ it reduces to the usual co-product action of $\mathfrak{g}$ on the product space

$$a \to \Delta(a) = \sum \mathbb{I} \otimes \cdots \otimes a \otimes \cdots \otimes \mathbb{I}. \tag{6.8}$$

If $|\lambda_i\rangle$ are the highest weight (and hence cyclic) vectors for the representations V_{λ_i}, then, with the usual co-product action of $\mathfrak{g}$, the vector $|v\rangle \equiv |\lambda_1\rangle \otimes \cdots \otimes |\lambda_N\rangle$ is the highest weight vector for only one of the irreducible representations occurring in the decomposition of the tensor product of the V_{λ_i}. Under the action of $\mathfrak{g}[t]$, however, and provided that the ζ_i are all distinct, $|v\rangle$ is a cyclic vector for the entire tensor product space. This is because the matrix ζ_i^j is invertible, and so any vector $|u_1\rangle \otimes \cdots \otimes a|u_j\rangle \otimes \cdots \otimes |u_N\rangle$ is obtainable as a linear combination of $(a \otimes t^m)(|u_1\rangle \otimes \cdots \otimes |u_N\rangle)$ for different m, and any vector $|u_i\rangle$ in each of the V_{λ_i} is obtainable by the action of suitable a on $|\lambda_i\rangle$.

The Lie algebra $\mathfrak{g}[t]$ is graded by the degree of the polynomial in t. This grading extends to the universal enveloping algebra $U(\mathfrak{g}[t])$ and gives rise to a *filtration*—a nested set of vector spaces—

$$F^0 \subseteq \cdots \subseteq F^i \subseteq F^{i+1} \subseteq \cdots \tag{6.9}$$

where

$$F^i = U^{\leqslant i}(\mathfrak{g}[t])|v\rangle. \tag{6.10}$$

Here, $U^{\leqslant i}(\mathfrak{g}[t])$ denotes the elements of the universal enveloping algebra $U(\mathfrak{g}[t])$, which have degree in t less than or equal to i. The action of $\mathfrak{g}$, considered as the zero-grade component of $\mathfrak{g}[t]$, preserves this filtration and so has a well-defined action on each of the components of the associated graded spaces $\mathrm{gr}^i[F] \equiv F^i/F^{i-1}$. Each of the irreducible representations in the tensor product space will appear in one of these graded subspaces.

As an illustration, consider the simplest case $\mathfrak{g} = su(2)$ where

$$[e, f] = h, \qquad [h, e] = +2e, \qquad [h, f] = -2f. \tag{6.11}$$

In the usual physics spin-j notation, we have the decomposition $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$, or in our Dynkin index language

$$V_1 \otimes V_1 = V_2 \oplus V_0. \tag{6.12}$$

The repeated action of the step-down operator f on the state

$$|v\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \tag{6.13}$$

(where $|\uparrow\rangle$ and $|\downarrow\rangle$ denote the states of weight 1 and -1, respectively) generates only the three states appearing in the spin-1 representation V_2, which therefore constitutes the space F^0. The action of $f \otimes t$, however, yields

$$\begin{aligned}(f \otimes t)|v\rangle &= \zeta_1|\downarrow\rangle \otimes |\uparrow\rangle + \zeta_2|\uparrow\rangle \otimes |\downarrow\rangle, \\ &= \tfrac{1}{2}(\zeta_1 - \zeta_2)(|\downarrow\rangle \otimes |\uparrow\rangle - |\uparrow\rangle \otimes |\downarrow\rangle) + \tfrac{1}{2}(\zeta_1 + \zeta_2)(|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle),\end{aligned} \tag{6.14}$$

which, since $|\downarrow\rangle \otimes |\uparrow\rangle + |\uparrow\rangle \otimes |\downarrow\rangle$ lies in F_0, is equivalent in F^1/F^0 to

$$\tfrac{1}{2}(\zeta_1 - \zeta_2)(|\downarrow\rangle \otimes |\uparrow\rangle - |\uparrow\rangle \otimes |\downarrow\rangle). \tag{6.15}$$

This last vector is the highest weight (indeed the only weight) in the spin-0 representation.

While the highest weight vectors of each representation occurring in the tensor product must appear somewhere in this construction, their coefficients depend quite non-trivially on the ζ_i, and some of these coefficients might vanish at non-generic points even for non-coincident ζ_i. It is therefore not obvious that the grade at which a given representation first appears is independent of the choice of the ζ_i. Feigin and Loktev conjecture that this is the case, and if this is true, the graded character

$$\mathrm{ch}_q\left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}\right) = \sum_d q^d \mathrm{ch}(\mathrm{gr}^d[F]) \tag{6.16}$$

should be independent of the ζ_i. In [5], we provide a proof of this conjecture by Feigin and Loktev in the spacial case where $\mathfrak{g} = su(r+1)$ and when the λ_i correspond to rectangular representations.

In the $su(2)$ example the ζ_i independence is manifest, and we have

$$\mathrm{ch}_q(V_1 \otimes V_1) = \mathrm{ch}\, V_2 + q\, \mathrm{ch}\, V_0. \tag{6.17}$$

Applying this construction to $su(4)$ we would find

$$\mathrm{ch}_q(V_{0,0,1} \otimes V_{0,1,0} \otimes V_{1,0,0}) = \mathrm{ch}\, V_{1,1,1} + (q+q^2)\,\mathrm{ch}\, V_{0,1,0} + q\,\mathrm{ch}\, V_{2,0,0} + q\,\mathrm{ch}\, V_{0,0,2}, \tag{6.18}$$

which looks like (6.3), but with $q \leftrightarrow 1/q$ to reflect the fact that Feigin and Loktev define their grading in the opposite direction to the one which is natural for the affine algebra.

To obtain the characters for arbitrary representations of $\widehat{su}(r+1)$, we only need to consider the fusion product of r rectangular representations $V_1, \ldots, V_r$, such that the highest weight λ_i of V_i is given by $\lambda_i = n_i\omega_i$, with $n_i \geqslant 0$. We will denote the q-polynomial coefficients in the decomposition of the character of this fusion product by $\mathcal{K}_{\mathbf{l},\mathbf{n}}(q)$. Here, $\mathbf{l}$ is the vector whose entries are the Dynkin indices of the representations present in the fusion product. The entries of $\mathbf{n}$ are the n_i. In appendix B, we show how these q-polynomials can be calculated. The result is the fermionic formula (B.6). Details can be found in our paper [5], where we also showed that the q-polynomials are in fact generalized Kostka polynomials [6, 7]. Note that we will always assume that $\sum_i n_i \leqslant k$, so we do not impose the level-restriction conditions in calculating the Kostka polynomials[2]. This restriction is always met in our case of obtaining characters of general $\widehat{su}(r+1)$ representations.

Thus, we have the following expression for the character of the fusion product of rectangular representations

$$\mathrm{ch}_q(V_1 \otimes V_2 \otimes \cdots \otimes V_r) = \sum_{\mathbf{l}} \mathcal{K}_{\mathbf{l},\mathbf{n}}(q)\,\mathrm{ch}\, V_{\mathbf{l}}. \tag{6.19}$$

These q-polynomial $\mathcal{K}_{\mathbf{l},\mathbf{n}}(q)$ coefficients that reduce for $q = 1$ to the Littlewood–Richardson coefficients are generalized Kostka polynomials. As a result, we conclude that the sum in equation (6.19) is finite.

Classical Kostka polynomials are parametrized by two Young diagrams, λ and μ, such that $|\lambda| = |\mu|$. Generalized Kostka polynomials [6], which we consider in this paper, are parametrized by a Young diagram λ and a set of rectangular diagrams $\{\mu_i\}$, such that the total number of boxes in the set $\{\mu_i\}$ is the same as the number of boxes in λ. Note that these Young diagrams are associated to $\mathfrak{gl}_{r+1}$. In our notation used above, the Young diagrams are associated to $\mathfrak{sl}_{r+1}$, which can be obtained from the $\mathfrak{gl}_{r+1}$ diagrams by 'stripping off' the columns of height $r+1$ (see [5] for more details).

The classical Kostka polynomial is a special case of the generalized Kostka polynomial, where the set $\{\mu_i\}$ case is the set of single-row diagrams, each equal to a single row of the diagram μ.

In both cases, the integer $K_{\lambda,\{\mu_i\}}(1)$ is equal to the multiplicity of the $\mathfrak{gl}_n$-module V_λ in the tensor product of representations $V_{\mu_1} \otimes \cdots \otimes V_{\mu_N}$. The Kostka polynomial is therefore a refinement of tensor product multiplicities, or a grading.

7. Wess–Zumino–Witten conformal theory

Although the geometric, or fusion, co-product and the evaluation representation were initially defined for the finite-dimensional Lie algebra $\mathfrak{g}$, they are motivated by the action of the affine Lie algebra $\widehat{\mathfrak{g}}$ in Wess–Zumino–Witten (WZW) conformal field theory [19].

We will focus on the holomorphic half of a level-k WZW model defined on the Riemann sphere. The symmetry algebra of this model is $\mathfrak{g} \otimes \mathcal{M}(\boldsymbol{\zeta})$, consisting of $\mathfrak{g}$-valued meromorphic functions with possible poles at the points ζ_i (where $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_N)$).

Let λ denote the highest weight of a finite-dimensional representation V_λ of the finite-dimensional algebra $\mathfrak{g}$, and suppose that the corresponding (in that its top graded component

[2] Level-restricted generalized Kostka polynomials [18] are also important, but do not play a role in our calculations.

is V_λ) representation $H_{\widehat{\lambda}}$ of the level-k affine Lie algebra $\widehat{\mathfrak{g}}$ is integrable. Then, the Wess–Zumino primary field $\varphi_\lambda(\zeta)$ acts on the vacuum at the origin $\zeta = 0$ there to create the highest weight state $|\widehat{\lambda}\rangle = \varphi_\lambda(0)|0\rangle$ of $H_{\widehat{\lambda}}$. If

$$a(z) = \sum_{n=-\infty}^{\infty} a[n] z^{-n-1} \tag{7.1}$$

is the WZW current associated with the element $a \in \mathfrak{g}$, and γ is any contour surrounding the origin then

$$a[n] = \frac{1}{2\pi \mathrm{i}} \oint_\gamma z^n a(z)\,\mathrm{d}z \tag{7.2}$$

has its usual action as an element of $\widehat{\mathfrak{g}}$ on this representation.

More generally, for integrable $\widehat{\lambda}_i$ we create highest weight states $|\widehat{\lambda}_i\rangle_{\zeta_i}$ at the points ζ_i by acting on the vacuum at these points with $\varphi_{\lambda_i}(\zeta_i)$ on the vacuum at ζ_i. The current algebra acts on the tensor product $H_{\widehat{\lambda}_1} \otimes \cdots \otimes H_{\widehat{\lambda}_N}$ of these spaces. We recall how this comes about. Let $f(t)$ denote a function meromorphic on the Riemann sphere, and having poles at no more than the ζ_i. If Γ is a contour surrounding $\zeta_1, \ldots, \zeta_N$, then we insert $\frac{1}{2\pi i}\oint_\Gamma f(t)a(t)\,\mathrm{d}t$ into a suitable correlation function. The contour can be deformed to a sum of contours γ_i each enclosing only one of the ζ_i, and the $a[n]$ acting on the representation space at ζ_i are the local mode-expansion coefficients $a(z) = \sum_n a[n](z - \zeta_i)^{-n-1}$, or

$$a[n] = \frac{1}{2\pi \mathrm{i}} \oint (t - \zeta_i)^n a(t)\,\mathrm{d}t. \tag{7.3}$$

Consequently, if

$$f(t) = \sum_{n=-p_i}^{\infty} f_n(\zeta_i)(t - \zeta_i)^n \tag{7.4}$$

is the Laurent expansion of f about ζ_i then $a \otimes f$ acts on a state $|v\rangle_{\zeta_i} \in H_{\lambda_i}$ as

$$\sum_{n=-p_i}^{\infty} f_n(\zeta_i) a[n] |v\rangle_{\zeta_i}, \tag{7.5}$$

and on $H_{\widehat{\lambda}_1} \otimes \cdots \otimes H_{\widehat{\lambda}_N}$ as

$$\Delta_{\zeta_1,\ldots,\zeta_N}(a \otimes f)(|v_1\rangle_{\zeta_1} \otimes \cdots \otimes |v_N\rangle_{\zeta_N})$$
$$= \sum_i |v_1\rangle_{\zeta_1} \otimes \cdots \otimes \left(\sum_{n=-p_i}^{\infty} f_n(\zeta_i) a[n] |v_i\rangle_{\zeta_i} \right) \otimes \cdots \otimes |v_N\rangle_{\zeta_N}, \tag{7.6}$$

where $\Delta_{\zeta_1,\ldots,\zeta_N}$ is the 'geometric' co-product [20]

$$\Delta_{\zeta_1,\ldots,\zeta_N}(a \otimes f) = \sum_i \mathbb{I} \otimes \cdots \otimes \left(\sum_{n=-p_i} f_n(\zeta_i) a[n] \right) \otimes \cdots \otimes \mathbb{I}. \tag{7.7}$$

This co-product makes the tensor products of any number of level-k representations into a level-k representation. With the conventional co-product, the level would be Nk.

In the particular case that $f(t) = t^n$, the Laurent expansion about ζ_i is

$$t^n = \sum_{m=0}^{n} \binom{n}{m} \zeta_i^{n-m} (t - \zeta_i)^m. \tag{7.8}$$

If $|v\rangle_{\zeta_i}$ is a top-component state on which $a[n]$ with $n > 0$ acts as zero, the action on $|v\rangle_{\zeta_i}$ is as $\zeta_i^n a[0]|v\rangle_{\zeta_i}$. Since the zero-grade elements of $\widehat{\mathfrak{g}}$ form an algebra isomorphic to $\mathfrak{g}$, we recognize our evaluation co-product action [16] described in equation (6.7).

When the contour Γ encloses all insertions of φ_λ's on the Riemann sphere, it can be contracted away, and the algebra acts trivially. This observation motivates the definition of the space $\mathcal{H}(\zeta, \{\lambda\})$ of *conformal blocks*. A conformal block $\Psi \in \mathcal{H}(\zeta; \{\lambda\})$ is a mapping

$$\Psi : H_{\widehat{\lambda}_1} \otimes \cdots \otimes H_{\widehat{\lambda}_N} \to \mathbb{C}, \tag{7.9}$$

invariant under the action of any $a \otimes f, a \in \mathfrak{g}$, with f having poles at no more that the ζ_i. It may be shown [21] that such a mapping is uniquely specified by its evaluation on the top graded component $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}$, and that the dimension $\dim(\mathcal{H}(\zeta, \{\lambda\}))$ of the space of conformal blocks is the multiplicity of the identity operator in the fusion product of the φ_{λ_i}.

This definition coincides with the physicist's picture where

$$\Psi(\zeta_1, \ldots, \zeta_N) \equiv \Psi\left(|\mu_1\rangle_{\zeta_1} \otimes \cdots \otimes |\mu_N\rangle_{\zeta_N}\right) = \langle 0|R\left\{\varphi_{\mu_1}(\zeta_1) \ldots \varphi_{\mu_N}(\zeta_N)\right\}|0\rangle, \tag{7.10}$$

with μ_i weights in V_{λ_i}, is thought of as a radial-ordered correlation function of the primary fields φ_{μ_i}. This correlator is not uniquely defined because the chiral primary fields are not mutually local. It is, however, a solution of the Knizhnik–Zamolodchikov equations, whose solution space is precisely the space of conformal blocks as identified above. In the operator language this comes about because the $\langle 0|$ could be the dual of any of the copies of the highest weight state of the vacuum representation that occur in the fusion-product.

We now recognize what it is that our function-space character is counting. It is the number of linear independent functions of the form

$$\begin{aligned} F\left(\left\{z_j^{(i)}\right\}\right) &= \langle v|R\left\{f_{\alpha_1}\left(z_1^{(1)}\right)\cdots f_{\alpha_r}\left(z_{m^{(r)}}^{(r)}\right)\varphi_{\lambda_1}(\zeta_1)\cdots\varphi_{\lambda_r}(\zeta_r)\right\}|0\rangle, \\ &\equiv \sum \Psi\left(|v^*\rangle_\infty \otimes |v_1\rangle_{\zeta_1} \otimes \cdots \otimes |v_N\rangle_{\zeta_N}\right). \end{aligned} \tag{7.11}$$

Here the $\varphi_{\lambda_i}(\zeta_i)$ create rectangular representations with the highest weight $\lambda_i = l_i\omega_i$ (no sum on i) by acting on the vacuum at the points ζ_i. The state $\langle v|$, analogous to the $\langle v|$ of (3.1), is the dual of $|v\rangle$, which lies in a weight space in one of the integrable representations $H_{\widehat{\lambda}}$ occurring in the fusion product. The sum is over the $|v_j\rangle$ produced by the action of the f_α on the $|\widehat{\lambda}_i\rangle$. The state $|v^*\rangle_\infty$ is in the dual representation $H^*_{\widehat{\lambda}}$ that is located at the point ∞ on the Riemann sphere. The (left action) $\mathfrak{g}$ weight of $|v^*\rangle$ is minus that of $|v\rangle$. (The correspondence $\langle v| \leftrightarrow |v^*\rangle_\infty$ is explained in [21].)

The fusion product of primary fields in the WZW model coincides with the finite-dimensional algebra Littlewood–Richardson rules only when k is infinite. For finite k we must use the level-restricted fusion rules of the Verlinde algebra. The level restrictions have no effect, however, when we build up an integrable representation of $\widehat{\mathfrak{g}}$ by concatenating integrable rectangular representations, as we are doing in this paper. The representations appearing in our WZW fusion product therefore coincide with those appearing in the $\mathfrak{g}[t]$ fusion product, which in turn are given by the Littlewood–Richardson rules of $\mathfrak{g}$.

The fusion product itself is not a graded space. However, we can define a filtration on this space, in the same way as in the Feigin–Loktev fusion product. This filtration is inherited from the filtration of the algebra $U(\mathfrak{n}_- \otimes \mathbb{C}(t^{-1}))$, where we assign a degree zero to the product of primary fields. The associated graded space has the character (4.4), which we can regard as the character of the fusion product. It is a sum over irreducible characters with a certain shift in the overall degree. We can regard this as the statement that the vector $\langle v|$ in the matrix element belongs to a highest weight representation, with a highest weight on which d acts by some negative integer instead of 0. (For an alternative point of view, where the space of highest weight vectors is interpreted as a quotient of the integrable representation—which is naturally graded—see [22].)

We would like to stress again that the characters of this space do not depend on the locations ζ_i of the rectangular representations!

8. Characters for arbitrary $\widehat{su}(r+1)_k$ representations

In this section, we will combine the results of the previous section and give explicit character formulae for arbitrary (integrable) representations of $\widehat{su}(r+1)_k$. There is one more ingredient needed to do this, namely an explicit formula for the (generalized) Kostka polynomials, which form an essential ingredient in the character formulae, as explained before. In this section, we will give the character formulae in terms of the generalized Kostka polynomials. How to obtain an explicit formula will be described in appendix B (we refer to [5] for the details). The resulting formula for the generalized Kostka polynomials is stated in equation (B.6).

We can write the decomposition of the character $\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x})$ in the following way:

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x}) = \sum_{\mathbf{l}} \mathcal{K}_{\mathbf{l},\mathbf{n}}\left(\frac{1}{q}\right) \mathrm{ch}\,H_{\mathbf{l};k}(q,\mathbf{x}). \tag{8.1}$$

We need to make a few remarks about the sum in the decomposition (8.1). First of all, we would like to note that this sum is finite. To show this, we introduce the notion of the threshold level, which is the lowest level for which a highest weight λ corresponds to a highest weight representation. We will denote this threshold level by $k(\mathbf{l})$. For $\widehat{su}(r+1)$, it is simply given by $k(\mathbf{l}) = \sum_{i=1}^{r} l_i$. The only $\mathbf{l}$ for which the Kostka polynomial $\mathcal{K}_{\mathbf{l},\mathbf{n}}(q)$ is non-zero are the $\mathbf{l}$ such that $k(\mathbf{l}) \leqslant k(\mathbf{n})$. The only non-zero $\mathcal{K}_{\mathbf{l},\mathbf{n}}(q)$ with $k(\mathbf{l}) = k(\mathbf{n})$ is when $\mathbf{l} = \mathbf{n}$, in which case $\mathcal{K}_{\mathbf{n},\mathbf{n}}(q) = 1$, see equation (B.7). Using these results, we can view the Kostka polynomials $\mathcal{K}_{\mathbf{n},\mathbf{l}}(q)$ as elements of a square matrix $\mathbf{K}$, which is upper triangular and 1's on the diagonal. Hence, this matrix is invertible, if we order the 'highest weights' $\mathbf{l}$ according to the increasing threshold level. Note that we did not specify an ordering for weights with the same threshold level, but any ordering of those weights gives an upper triangular matrix.

Now we established that the Kostka matrix $\mathbf{K}$ is invertible, we can invert the relation (8.1) to obtain explicit character formulae for arbitrary (integrable) highest weight representations of $\widehat{su}(r+1)_k$

$$\mathrm{ch}\,H_{\mathbf{l};k}(q,\mathbf{x}) = \sum_{\mathbf{n}} (\mathbf{K}^{-1})_{\mathbf{n},\mathbf{l}}\left(\frac{1}{q}\right) \mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x}). \tag{8.2}$$

Again, the (finite) sum is over the highest weights $\mathbf{n}$ with the threshold level $k(\mathbf{n}) \leqslant k(\mathbf{l})$.

9. Conclusion

We have shown how the introduction of the fusion product representation allows us to generalize the Feigin–Stoyanovsky strategy for computing characters of affine $\widehat{su}(r+1)$ to all integrable representations. We found that for non-rectangular representations, the character is not of fermionic type, but can be written as a linear combination of fermionic characters. The coefficients are polynomials in $1/q$ and are related to the generalized Kostka polynomials.

It is rather straightforward to generalize the Feigin–Stoyanovsky construction to arbitrary affine Lie algebras. In that case, even characters of rectangular representations are not always of fermionic type. It turns out that the character of those representations can be written in terms of characters of the so-called 'Kirillov–Reshetikhin' modules [23]. Details will be provided in a forthcoming publication.

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Appendix A. The structure of the character of the principal subspace

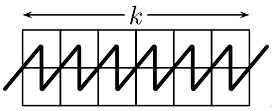
In this appendix, we will explain the structure of the character of the principal subspace of rectangular representations of $\widehat{su}(r+1)$, equation (3.13). To this end, we will explain the zeros of the function $f(z)$ which appears in (3.6).

The exponent of q in the character is the total degree of the polynomials $f\left(\left\{z_j^{(i)}\right\}\right)$, which ensure that the vanishing conditions hold, combined with the explicit poles in $F\left(\left\{z_j^{(i)}\right\}\right)$. The strategy presented in [1] and [5], which we will outline here briefly, is to find the minimal amount of zeros necessary to enforce the vanishing conditions.

Let us first focus on the case where we have $2k$ variables. We need to find a polynomial which does not vanish when k variables are set to the same location, but does vanish when $k+1$ variables are set to the same location. Of course, the condition (3.8) allows for functions which vanish when less than $k+1$ variables are set to the same location, but we will deal with those later. A symmetric polynomial in the variables $z_1, z_2, \ldots, z_{2k}$ with the lowest possible degree which satisfies this property is ($\mathcal{S}$ denotes symmetrization over all variables)

$$\mathcal{S}[(z_1 - z_{k+1})(z_2 - z_{k+1})(z_2 - z_{k+2}) \cdots (z_k - z_{2k-1})(z_k - z_{2k})(z_1 - z_{2k})]. \tag{A.1}$$

We can display these zeros nicely in a graphical way in terms of a Young diagram. The boxes on the top row correspond to the variables $z_1, \ldots, z_k$ while the boxes on the second row correspond to $z_{k+1}, \ldots, z_{2k}$. A line connecting the boxes of z_i and z_j corresponds to the factor $(z_i - z_j)$:



,

where 'periodic boundary conditions' are assumed, to obtain the zero $(z_1 - z_{2k})$. It easily follows that if we pick k variables, and set all of them to the same value, there will always be a term in the symmetrization of equation (A.1) which is not zero. However, setting $k+1$ variables to the same location gives a zero for each term in the symmetrization.

We will now discuss the general case, i.e. we allow for functions which vanish when fewer than $k+1$ variables are set equal, and we also allow for a general number of variables.

To do this, we need to label the polynomials by r partitions (or Young diagrams), one for each type of variable. These are partitions of the integers $m^{(i)}$, with the property that the width of the rows is maximally k, because the polynomials should vanish when $k+1$ variables are set to the same value.

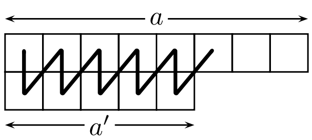
The number of rows of width a of partition (i) is given by $m_a^{(i)}$, i.e. the partitions have the form $\left(k^{m_k^{(i)}}, (k-1)^{m_{k-1}^{(i)}}, \ldots, 2^{m_2^{(i)}}, 1^{m_1^{(i)}}\right)$. It follows that we have the relation $m^{(i)} = \sum_{a=1}^{k} a m_a^{(i)}$. See figure 1 for an example where $m^{(1)} = 13$.

To each box of the partition (i), we associate a variable $z_j^{(i)}$. Let $z_1, \ldots, z_a$ be the variables associated to a row of length a, of partition (i) and $\bar{z}_1, \ldots, \bar{z}_{a'}$ to a row of length a', also from partition (i), such that $a' \leqslant a$.

We need to assign the following zero corresponding to these variables

$$(z_1 - \bar{z}_1)(z_2 - \bar{z}_1)(z_2 - \bar{z}_2)(z_3 - \bar{z}_2) \cdots (z_{a'} - \bar{z}_{a'})(z_{a'+1} - \bar{z}_{a'}).$$

We will identify $z_{a+1} = z_1$. Pictorially, we can represent these zeros as

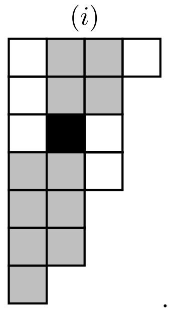


.

Again, each box corresponds to a variable, and a line connecting two boxes indicates a zero when the two corresponding variables have the same value. We obtain all the zeros needed to satisfy the integrability condition by multiplying all the zeros associated to each pair of rows belonging to the same partition

$$\prod_{i=1}^{r} \prod_{\substack{\text{pairs of rows}\\ \text{of partition } (i)\\ \text{of length } a,a'\\ a\geqslant a'}} \prod_{j=1}^{a'} \left(z_j^{(i)} - \bar{z}_j^{(i)}\right)\left(z_{j+1}^{(i)} - \bar{z}_j^{(i)}\right). \tag{A.2}$$

To clarify this, we will look at one particular variable (which will correspond to the black box in the diagram below), and show with which other variables it has a zero. These other variables are given by the grey boxes.



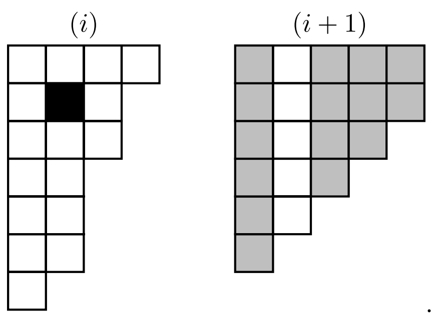
. (A.3)

To obtain all the zeros, we need to take the product over all 'black boxes', but without double counting the zeros. Finally, we need to take the product over all partitions $i = 1, \ldots, r$. It is not too hard to convince oneself that these zeros do imply the integrability condition.

We will now focus on the zeros we have to include in the polynomials $f\left(\left\{\left(z_j^{(i)}\right\}\right)$ to ensure that the Serre relations are taken into account properly. Let $z_j^{(i)}$ be a variable associated to a row of length a of the partition of $m^{(i)}$. This variable has zeros with variables $z_{j'}^{(i+1)}$ which belong to a row of the partition of $m^{(i+1)}$ of length a'. In particular, these zeros are $\prod_{j'\neq j}\left(z_j^{(i)} - z_{j'}^{(i+1)}\right)$. We obtain all the zeros associated to the pair of partitions (i) and $(i+1)$ by multiplying all the zeros associated to each variable of partition (i).

$$\prod_{i=1}^{r-1} \prod_{\substack{\text{rows of}\\ \text{partition } (i)}} \prod_{\substack{\text{rows of}\\ \text{partition } (i+1)}} \prod_{\substack{j,j'\\ j'\neq j}} \left(z_j^{(i)} - z_{j'}^{(i+1)}\right). \tag{A.4}$$

Pictorially, these zeros are given by



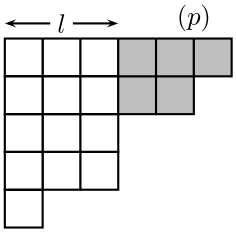
. (A.5)

Again, we need to take the product over all 'black boxes' and over $i = 1, \ldots, r-1$. Again, it is easy to check that these zeros, combined with the zeros which enforce integrability, give rise to the Serre conditions.

Finally, we need to include the zeros to make sure the highest weight condition is satisfied. For a row of length $a > l$ of the partition of $m^{(p)}$, we need to include the zeros

$$\prod_{\substack{\text{rows of} \\ \text{partition } p}} \prod_{j>l} z_j^{(p)}, \tag{A.6}$$

or, pictorially, the zeros are given by the grey boxes



$$. \tag{A.7}$$

Counting the total degree of the zeros implied by (A.3), (A.5) and (A.7), combined with the degree corresponding to the poles in (3.6) and the extra term $\sum_{i=1}^{r} m^{(i)}$ in the definition (3.11) precisely give the exponent of q in the character formula (3.13).

Apart from the zeros discussed above, $f\left(\left\{z_j^{(i)}\right\}\right)$ can contain additional symmetric polynomials. The degree of these symmetric polynomials is taken into account by the factors $\frac{1}{(q)_m}$ in (3.13) (see [5] for the details). The factor $\frac{1}{(q)_m}$ is the generating function for the symmetric polynomials in m variables. The coefficient of q^d in the expansion of $\frac{1}{(q)_m}$ is the number of symmetric polynomials of degree d in m variables.

Appendix B. Obtaining the Kostka polynomials

In the main text, we explained how the characters of the fusion product of rectangular representations can be decomposed into characters of arbitrary highest weight representations. In this appendix we will outline a strategy to obtain an explicit expression for the 'expansion coefficients', which turn out to be generalized Kostka polynomials. We refer to our paper [5] for the details of the proof.

We need to consider matrix elements of the form

$$G_{\lambda,\mu}(\zeta) = \langle u_\lambda | f_{\alpha_1}\left(z_1^{(1)}\right) \cdots f_{\alpha_r}\left(z_{m^{(r)}}^{(r)}\right) v_1(\zeta_1) \otimes \cdots \otimes v_r(\zeta_r), \tag{B.1}$$

where $\langle u_\lambda|$ is dual to the state $|u_\lambda^*\rangle_\infty$, as explained in the discussion following equation (7.11). The $v_i(\zeta_i)$ are the highest weights of the rectangular representations $H_{n_i\omega_i}(\zeta_i)$. Thus, the rectangular representations inserted at the points ζ_i are given by $n_i\omega_i$. We define $\mathbf{n} = (n_1, \ldots, n_r)^T$. In addition, we define $\boldsymbol{\mu} = \sum_i n_i\omega_i$ and the coefficients l_i are determined by $\lambda = \sum_i l_i\omega_i$.

To obtain a formula for the generalized Kostka polynomials, we should only consider the bra's $\langle u_\lambda|$ at infinity, which are the highest weights of the 'bottom component' of the representation located at infinity. Note that the representation which is located at infinity is turned 'upside down', because we chose $\frac{1}{z}$ as the local variable at infinity.

We need to find the conditions such that the matrix elements (B.1) are non-zero. First of all, we need that $\lambda = \boldsymbol{\mu} - \sum_i m^{(i)}\alpha_i$, which translates into

$$\mathbf{m} = C_r^{-1} \cdot (\mathbf{n} - \mathbf{l}). \tag{B.2}$$

As usual, we also need to impose the highest weight conditions $f_{\alpha_i}^{n_i+1}[0]|v_i\rangle = 0$ and the Serre relations. Finally, we need to incorporate a new ingredient, which is a degree restriction on the functions in the function space. These degree restrictions can be derived by letting the f's in the matrix elements (B.1) act to the left on u_λ. Now, $f_\alpha[n]$ acts trivially at ∞ if $n \leqslant 0$, which translates a degree restriction on the functions.

The space of functions which we need to consider is spanned by rational functions in the variables $z_j^{(i)}$, which are symmetric under the exchange $z_j^{(i)} \leftrightarrow z_{j'}^{(i)}$. There can be poles of order 1 at the positions $z_j^{(i)} = z_{j'}^{(i+1)}$ and at $z_j^{(i)} = \zeta_i$; that is,

$$G(\mathbf{z}) = \frac{g(\mathbf{z})}{\prod_{i=1}^{r}\prod_j \left(\zeta_i - z_j^{(i)}\right)\prod_{i=1}^{r-1}\prod_{j,j'}\left(z_j^{(i)} - z_{j'}^{(i+1)}\right)}. \tag{B.3}$$

The polynomial $g(\mathbf{z})$ (which is symmetric under the exchange of variables of the same colour) vanishes when any of the following conditions holds:

$$z_1^{(i)} = z_2^{(i)} = z_1^{(i+1)}, \qquad z_1^{(i)} = z_1^{(i+1)} = z_2^{(i+1)}, \qquad i = 1, \dots, r-1, \tag{B.4}$$

$$z_1^{(i)} = z_2^{(i)} = \cdots = z_{l_i+1}^{(i)} = \zeta_i, \qquad \forall i. \tag{B.5}$$

Note that we do not impose the integrability conditions, because k will always be large enough, i.e. $k \geqslant \sum_i n^{(i)}$. Finally, we need to impose the degree restriction on the function $G(\mathbf{z})$. For each variable, we have that $\deg_{z_j^{(i)}} \leqslant 2$, which follows from the fact that we use $\frac{1}{z}$ as a local variable at infinity. In [5], we showed that this leads to the following polynomials:

$$\mathcal{K}_{\mathbf{l};\mathbf{n}}(q) = \sum_{\substack{m_a^{(i)}\geqslant 0\\ i=1,\dots,r\\ \mathbf{m}=(\mathbf{C}_r^{-1})\cdot(\mathbf{n}-\mathbf{l})}} q^{\frac{1}{2}m_a^{(i)}(\mathbf{C}_r)_{i,j}\mathbf{A}_{a,b}m_b^{(j)}} \prod_{i,a}\begin{bmatrix}\mathbf{A}_{a,n^{(i)}} - (\mathbf{C}_r)_{i,j}\mathbf{A}_{a,b}m_b^{(j)} + m_a^{(i)}\\ m_a^{(i)}\end{bmatrix}_q, \tag{B.6}$$

where $[\,]_q$ denotes the q-binomial, which is defined to be $\begin{bmatrix}n+m\\ m\end{bmatrix}_q = \frac{(q)_{n+m}}{(q)_n(q)_m}$ for $m, n \in \mathbb{Z}_{\geqslant 0}$ and zero otherwise. The coefficient of q^d in the expansion of $\begin{bmatrix}n+m\\ m\end{bmatrix}_q$ is the number of symmetric polynomials of total degree d, in m variables, where the degree of each variable is maximally n. These q-binomials arise because of the degree restrictions on the polynomials $G(\mathbf{z})$. Note that the Kostka polynomials do not depend on the positions ζ_i, which follows from the fact that the degree of the rational functions (B.3) does not depend on these positions.

In [5], we showed that the functions $\mathcal{K}_{\mathbf{l};\mathbf{n}}(q)$ of (B.6) are polynomials in q, and are related in a simple way to the generalized Kostka polynomials of [6, 7]. In particular, by setting $q = 1$, we obtain the Littlewood–Richardson coefficients.

Let us make a few remarks about the structure of the polynomials $\mathcal{K}_{\mathbf{l};\mathbf{n}}(q)$. Let $k(\mathbf{l}) = \sum_i l_i$, which is the lowest level at which $\lambda = \sum_i l_i\omega_i$ corresponds to an integrable representation (i.e. $k(\mathbf{l})$ is the threshold level). We then have the following results:

$$\mathcal{K}_{\mathbf{l};\mathbf{l}}(q) = 1 \qquad \mathcal{K}_{\mathbf{l};\mathbf{n}}(q) = 0 \quad \text{if} \quad \begin{cases} k(\mathbf{l}) > k(\mathbf{n}) \\ k(\mathbf{l}) = k(\mathbf{n}) \text{ and } \mathbf{l} \neq \mathbf{n} \\ \sum_i i l_i \neq \sum_i i n^{(i)} \bmod r+1. \end{cases} \tag{B.7}$$

These results are obtained by making use of the constraint $\mathbf{m} = (\mathbf{C}_r)^{-1}(\mathbf{n}-\mathbf{l})$ and the fact that all the summation variables $\mathbf{m} = (m_1, \dots, m_r)$ have to be non-negative integers in the sum in equation (B.6).

We can view the polynomials $\mathcal{K}_{\mathbf{l};\mathbf{n}}$ as the entries of a (square) matrix $\mathbf{K}$, with entries $(\mathbf{K})_{\mathbf{l};\mathbf{n}}(q) = \mathcal{K}_{\mathbf{l};\mathbf{n}}(q)$. The relations (B.7) imply that there is an ordering such that the matrix $\mathbf{K}$ is upper triangular with 1s on the diagonal. Thus, the matrix $\mathbf{K}$ is invertible.

Before we move on and give the character formulae for arbitrary highest weight representations, we will first give an example of the 'Kostka matrix' related to $\widehat{su}(4)$. We use the following ordering of the entries $\mathbf{l}$:

(0, 0, 0); (1, 0, 1), (0, 2, 0); (2, 1, 0), (0, 1, 2); (4, 0, 0), (2, 0, 2), (1, 2, 1), (0, 4, 0), (0, 0, 4).

With this ordering, we obtain the following Kostka matrix

$$\mathbf{K}(q) = \left(\begin{array}{c|cc|cc|ccccc} 1 & q & 0 & 0 & 0 & 0 & q^2 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & q & q & 0 & q & q^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & q+q^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & q & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{B.8}$$

The inverse is

$$\mathbf{K}^{-1}(q) = \left(\begin{array}{c|cc|cc|ccccc} 1 & -q & 0 & q^2 & q^2 & 0 & 0 & -q^3 & 0 & 0 \\ \hline 0 & 1 & 0 & -q & -q & 0 & -q & q^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -q-q^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -q & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{B.9}$$

Using the results of sections 6 and 7, we can now write down explicit character formulae for arbitrary integrable highest weight representations of $\widehat{su}(r+1)$, in terms of the characters $\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x})$ and the inverse matrix $\mathbf{K}^{-1}\left(\frac{1}{q}\right)$.

First of all, we have the result that we can decompose the characters $\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x})$ in terms of the characters of arbitrary integrable highest weight representations $\mathrm{ch}\,H_{\mathbf{l};k}(q,\mathbf{x})$ as follows:

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x}) = \sum_{\substack{\mathbf{l} \\ k(\mathbf{l})\leqslant k(\mathbf{n})}} \mathcal{K}_{\mathbf{l};\mathbf{n}}\left(\frac{1}{q}\right) \mathrm{ch}\,H_{\mathbf{l};k}(q,\mathbf{x}). \tag{B.10}$$

We can invert this relation to obtain explicit character formulae for arbitrary integrable highest weight representations of $\widehat{su}(r+1)$

$$\mathrm{ch}\,H_{\mathbf{l};k}(q,\mathbf{x}) = \sum_{\substack{\mathbf{n} \\ k(\mathbf{n})\leqslant k(\mathbf{l})}} (\mathbf{K}^{-1})_{\mathbf{n};\mathbf{l}}\left(\frac{1}{q}\right) \mathrm{ch}\,\mathcal{F}^{\infty}_{\mathbf{n};k}(q,\mathbf{x}). \tag{B.11}$$

We would like to note that we have a similar formula for the character of the principal subspace of the general highest weight representations

$$\mathrm{ch}\,W_{\mathbf{l};k}(q,\mathbf{x}) = \sum_{\substack{\mathbf{n} \\ k(\mathbf{n})\leqslant k(\mathbf{l})}} (\mathbf{K}^{-1})_{\mathbf{n};\mathbf{l}}\left(\frac{1}{q}\right) \mathrm{ch}\,\mathcal{F}_{\mathbf{n};k}(q,\mathbf{x}), \tag{B.12}$$

where the character $\mathcal{F}_{\mathbf{l};k}(q,\mathbf{x})$ is given by

$$\mathrm{ch}\,\mathcal{F}_{\mathbf{l};k}(q,\mathbf{x}) \stackrel{\mathrm{def}}{=} \left(\prod_{i=1}^{r} x_i^{l_i}\right) \sum_{\substack{m_a^{(i)}\in\mathbb{Z}_{\geqslant 0}\\ a=1,\ldots,k\\ i=1,\ldots,r}} \left(\prod_{i=1}^{r} x_i^{-\sum_j (\mathbf{C}_r)_{ji} m^{(j)}}\right) \frac{q^{\frac{1}{2}m_a^{(i)}(\mathbf{C}_r)_{i,i'}\mathbf{A}_{a,a'}m_{a'}^{(i')}-\mathbf{A}_{a,l_i}m_a^{(i)}}}{\prod_{i=1}^{r}\prod_{a=1}^{k}(q)_{m_a^{(i)}}}. \tag{B.13}$$

This character can be viewed as the 'untranslated' version of the character (4.4), because applying the affine Weyl translation (as explained in section 4) results in (4.4).

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